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SIMULTANEOUS FACTOR ANALYSIS OF DICHOTOMOUS VARIABLES IN SEVERAL GROUPS

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A new method is proposed for a simultaneous factor analysis of dichotomous responses from several groups of individuals. The method makes it possible to compare factor loading pattern, factor variances and covariances, and factor means over groups. The method uses information from first and second order proportions and estimates the model by generalized least-squares. Hypotheses regarding different degrees of invariance over groups may be evaluated by a large-sample chi-square test.

Key words: group comparisons, invariant measurement parameters, factor means.

This article is concerned with factor analysis of dichotomous variables. Bock and Lieberman [1970] treated the maximum-likelihood method for the case of one common factor. This model was generalized to multiple factors by Christoffersson [1975] who proposed an alternative and somewhat simpler estimation technique using limited information. Muthén [1978] further simplified the estimation in the multiple factor case using the same amount of information as in Christoffersson [1975]. Common to these articles is the specification of latent continuous multivariate normal response variables underlying the observed dichotomous variables; one latent variable corresponding to each observed variable. Due to this fact, the approach of Muthén [1978] is similar to ordinary factor analysis of tetrachoric correlations [see Muthén, 1978, p. 555]. In Muthén [1978], however, it is recognized that the sample tetrachoric correlations have a different large sample covariance matrix than the Pearson correlations assumed in an ordinary factor analysis. Although consistent estimates can be obtained, ordinary factor analysis of tetrachoric correlations hence does not give correct standard errors of estimates or a correct chi-square test of model fit (see also Bock & Lieberman, 1970). It may also be noted that for the model of Muthén [1978] ordinary factor analysis of the Pearson correlations for the dichotomous variables (the phi coefficients) results in inconsistent and attenuated estimates in addition to incorrect standard errors of estimates and incorrect chi-square test of model fit (see Olsson, 1979). In the present article the approach of Muthén [1978] is further developed to cover a more general model.

Christoffersson [1975] noted the possible generalization of the model for the analysis of several groups of individuals simultaneously, in a way analogous to ordinary factor analysis of continuous variables (see Jöreskog, 1971, & Sörbom, 1974). This is of interest when different groups of people, for instance persons characterized by different socioeconomic backgrounds, are administered the same or a similar measurement instrument. The object is to study differences and similarities between groups. This may be done with respect to structural parameters, namely the factor means, factor variances and covariances,

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and the error variances, and also with respect to the measurement parameters (the item parameters).

Of particular interest is the case where the exact same instrument is administered to all groups, for one may then assume that the measurement parameters are invariant over groups. Under this assumption, the model proposed in this paper makes possible the identification and estimation of the group-specific factor means. To consider factor means that vary over groups is a new feature of factor analysis, which was developed for continuous observed variables by Sörbom [1974]. This is a useful way of studying group level differences, purged of the influence of measurement error in the observed variables [see also Jöreskog, 1981, & Sörbom, 1981]. For the case of dichotomous variables, the use of factor means provides an alternative to the common use of summed item scores (e.g., 0's and 1's) in comparisons between groups.

Since the variables defining the different groups may also be seen as categorical, the data to be analyzed can be described in a multi-way contingency table. A recently popularized way to study such data is by means of so called log linear models [see e.g., Haberman, 1978, 1979]. Differences and similarities between groups can be studied also with that technique but the disadvantage is that only relationships between observed variables are considered, whereas in our approach it is explicitly recognized that these variables are only fallible indicators of the latent variables of primary interest. There seems to be no obvious and direct relationship between these two analysis approaches.

The General Model

Let $\mathbf{x}^{(g)}$ denote a vector of p observed dichotomous variables for the g^{th} group, $g = 1, 2, \ldots, G$. Let $\boldsymbol{\tau}^{(g)}$ $(p \times 1)$ be a parameter vector of thresholds for a vector $\boldsymbol{\xi}^{*(g)}$ $(p \times 1)$ of latent response variables underlying $\mathbf{x}^{(g)}$. For each group g we assume, as in Christoffersson [1975] and Muthén [1978], that for $i = 1, 2, \ldots, p$

$$\mathbf{x}_i^{(g)} = \begin{cases} 1, & \text{if } \xi_i^{*(g)} > \tau_i^{(g)}, \\ 0, & \text{otherwise,} \end{cases} \tag{1}$$

$$\boldsymbol{\xi}^{*(g)} = \boldsymbol{\Lambda}^{(g)}\boldsymbol{\xi}^{(g)} + \boldsymbol{\varepsilon}^{(g)}, \tag{2}$$

where $\boldsymbol{\Lambda}^{(g)}(p \times k)$ is a matrix of factor loadings, $\boldsymbol{\xi}^{(g)}(k \times 1)$ is a vector of factor scores, and $\boldsymbol{\varepsilon}^{(g)}(p \times 1)$ is a vector of errors assumed to be uncorrelated with $\boldsymbol{\xi}^{(g)}$ and to have zero expectations. For each group, g, $\boldsymbol{\xi}^{*(g)}$ is assumed to be multivariate normal. The mean vector and covariance matrix of $\boldsymbol{\xi}^{(g)}$, and the covariance matrix of $\boldsymbol{\varepsilon}^{(g)}$ are allowed to vary over groups:

$$E(\boldsymbol{\xi}^{(g)}) = \mathbf{v}^{(g)}, \tag{3}$$

$$V(\boldsymbol{\xi}^{(g)}) = \boldsymbol{\Phi}^{(g)}, \tag{4}$$

$$V(\boldsymbol{\varepsilon}^{(g)}) = \boldsymbol{\psi}^{(g)}. \tag{5}$$

In the general case, no assumption of diagonality is made for the $\boldsymbol{\psi}^{(g)}$-matrices. This allows for correlated errors, which is a useful feature in some cases.

The model implies that

$$E(\boldsymbol{\xi}^{*(g)}) = \boldsymbol{\Lambda}^{(g)}\mathbf{v}^{(g)}, \tag{6}$$

and that

$$V(\boldsymbol{\xi}^{*(g)}) = \boldsymbol{\Lambda}^{(g)}\boldsymbol{\Phi}^{(g)}\boldsymbol{\Lambda}^{(g)\prime} + \boldsymbol{\psi}^{(g)} = \boldsymbol{\Sigma}^{(g)}, \text{ say.} \tag{7}$$

Let

$$\mathbf{D}^{(g)} = [\text{diag}(\mathbf{\Sigma}^{(g)})]^{-1/2}.$$

Now consider the vector of standardized latent response variables

$$\boldsymbol{\xi}^{**(g)} = \mathbf{D}^{(g)}(\boldsymbol{\xi}^{*(g)} - \mathbf{\Lambda}^{(g)}\mathbf{v}^{(g)}). \tag{8}$$

The following two arrays will be of particular interest.

$$\boldsymbol{\theta}_1^{(g)} = \mathbf{D}^{(g)}(\boldsymbol{\tau}^{(g)} - \mathbf{\Lambda}^{(g)}\mathbf{v}^{(g)}), \tag{9}$$

$$\mathbf{\Theta}_2^{(g)} = \mathbf{D}^{(g)}(\mathbf{\Lambda}^{(g)}\mathbf{\Phi}^{(g)}\mathbf{\Lambda}^{(g)\prime} + \boldsymbol{\psi}^{(g)})\mathbf{D}^{(g)}. \tag{10}$$

The vector $\boldsymbol{\theta}_1^{(g)}$ has p elements. Let $\boldsymbol{\theta}_2^{(g)}$ be the vector of the $p(p-1)/2$ off-diagonal elements of the correlation matrix $\mathbf{\Theta}_2^{(g)}$. The elements of $\boldsymbol{\theta}_1^{(g)}$ and $\mathbf{\Theta}_2^{(g)}$ determine the multinomial distribution for each $\mathbf{x}^{(g)}$ obtained by integration over $\boldsymbol{\xi}^{**(g)}$. In the estimation procedure to be described we will fit the first and second order marginal probabilities, deduced from the integrated standard normal densities:

$$P(x_i^{(g)} = 1) = \int_{\theta_i^{(g)}}^{\infty} \phi(\xi_i^{**(g)})\, d\xi_i^{**(g)}, \tag{11}$$

$$P(x_i^{(g)} = 1, x_j^{(g)} = 1) = \int_{\theta_i^{(g)}}^{\infty}\int_{\theta_j^{(g)}}^{\infty} \phi(\xi_i^{**(g)}, \xi_j^{**(g)}; \theta_{ij}^{(g)})\, d\xi_i^{**(g)}\, d\xi_j^{**(g)}, \tag{12}$$

where $\theta_i^{(g)}$ is the i^{th} element of $\boldsymbol{\theta}_1^{(g)}$, $\theta_{ij}^{(g)}$ is the i, j^{th} element (correlation) of $\mathbf{\Theta}_2^{(g)}$, $\phi(z)$ is the standard normal density function, and $\phi(z_1, z_2; \theta)$ is the bivariate standard normal density function with correlation θ.

The model is not identified without restrictions on the parameters. It is difficult to give general rules which are sufficient for identification and this will not be attempted here. However, from (9) and (10) we note two basic types of indeterminacies due to the fact that $\boldsymbol{\xi}^{*(g)}$ is not observed. Firstly, it is not possible to separately identify the two sets of location parameters $\boldsymbol{\tau}^{(g)}$ and $\mathbf{v}^{(g)}$. Secondly, the correlations but not the variances-covariances of $\boldsymbol{\xi}^{*(g)}$ are identified. The p variances of $\boldsymbol{\varepsilon}^{*(g)}$, contained in $\mathbf{D}^{(g)}$, are functions of the p free parameters in the diagonal of $\boldsymbol{\psi}^{(g)}$. Thereby, we cannot identify $\boldsymbol{\tau}^{(g)}$, $\mathbf{\Lambda}^{(g)}$ and the off-diagonal elements of $\boldsymbol{\psi}^{(g)}$, but only the product-forms $\mathbf{D}^{(g)}\boldsymbol{\tau}^{(g)}$, $\mathbf{D}^{(g)}\,\mathbf{\Lambda}^{(g)}$ and $\mathbf{D}^{(g)}\boldsymbol{\psi}^{(g)}\mathbf{D}^{(g)}$.

In the next section it will be shown how these indeterminacies can be removed. That section deals with a special case of the general model which is of particular interest and results in an identified model.

Parameters Invariant Over Groups

The parameters of $\boldsymbol{\tau}^{(g)}$ and $\mathbf{\Lambda}^{(g)}$ describe the measurement properties of the dichotomous variables (the item parameters). When the same instrument has been administrated to all groups it may therefore be reasonable to assume that these measurement parameters are invariant over groups, that is,

$$\boldsymbol{\tau}^{(1)} = \boldsymbol{\tau}^{(2)} = \cdots = \boldsymbol{\tau}^{(G)} = \boldsymbol{\tau}, \tag{13}$$

$$\mathbf{\Lambda}^{(1)} = \mathbf{\Lambda}^{(2)} = \cdots = \mathbf{\Lambda}^{(G)} = \mathbf{\Lambda}. \tag{14}$$

Although the thresholds and the loading matrix are constant over groups, the means, variances, and covariances of the latent response variables $\boldsymbol{\xi}^{*(g)}$ are allowed to vary with the parameters of $\mathbf{v}^{(g)}$, $\mathbf{\Phi}^{(g)}$ and $\boldsymbol{\psi}^{(g)}$.

In addition to (13) and (14) we may assume invariance of the error variances. With uncorrelated errors this is stated as,

$$\psi^{(1)} = \psi^{(2)} = \cdots = \psi^{(G)}. \tag{15}$$

In the single-factor case, (13), (14) and (15) correspond to invariance of the item characteristic curves of the normal ogive (see next section). Thus, we may alternatively view the error variances as also describing measurement characteristics of the items.

The different hypotheses (13), (14), (15), and many others, may all be tested by the procedure described in the section on estimation.

We will now consider the model given that (13) and (14) hold. The two types of indeterminacies mentioned above will be removed as follows. Consider an arbitrary group f. This will be used as a reference group by fixing the factor means to zero and the variances of the latent response variables to unity, $\mathbf{v}^{(f)} = \mathbf{0}$ and $\text{diag}(\boldsymbol{\Sigma}^{(f)}) = \mathbf{I}$. By (9) and (10),

$$\boldsymbol{\theta}_1^{(f)} = \boldsymbol{\tau}_f, \tag{16}$$

$$\boldsymbol{\Theta}_2^{(f)} = \boldsymbol{\Lambda}_f \boldsymbol{\Phi}_f^{(f)} \boldsymbol{\Lambda}_f' + \boldsymbol{\psi}_f^{(f)}, \tag{17}$$

where

$$\text{diag}(\boldsymbol{\psi}_f^{(f)}) = \mathbf{I} - \text{diag}(\boldsymbol{\Lambda}_f \boldsymbol{\Phi}_f^{(f)} \boldsymbol{\Lambda}_f'). \tag{18}$$

The subscript refers to the standardization of the parameters using group f as a reference group. To determine the scale of each factor we will fix one element in each column of $\boldsymbol{\Lambda}_f$ to a nonzero value (e.g., 1). When there is only one group ($G = 1$), this is the parameterization of Christoffersson [1975] and Muthén [1978]. For the other groups ($g \neq f$), $\mathbf{v}^{(g)}$ and $\text{diag}(\boldsymbol{\psi}^{(g)})$ are not constrained,

$$\boldsymbol{\theta}_1^{(g)} = \mathbf{D}_f^{(g)}(\boldsymbol{\tau}_f - \boldsymbol{\Lambda}_f \mathbf{v}_f^{(g)}), \tag{19}$$

$$\boldsymbol{\Theta}_2^{(g)} = \mathbf{D}_f^{(g)}(\boldsymbol{\Lambda}_f \boldsymbol{\Phi}_f^{(g)} \boldsymbol{\Lambda}_f' + \boldsymbol{\psi}_f^{(g)})\mathbf{D}_f^{(g)}, \tag{20}$$

$$\mathbf{D}_f^{(g)} = [\text{diag}(\boldsymbol{\Lambda}_f \boldsymbol{\Phi}_f^{(g)} \boldsymbol{\Lambda}_f' + \boldsymbol{\psi}_f^{(g)})]^{-1/2}. \tag{21}$$

Without changing the model we may transform the parameterization using reference group f to the parameterization using reference group h in the following way:

$$\boldsymbol{\tau}_h = \boldsymbol{\theta}_1^{(h)} = \mathbf{D}_f^{(h)}(\boldsymbol{\tau}_f - \boldsymbol{\Lambda}_f \mathbf{v}_f^{(h)}), \tag{22}$$

$$\boldsymbol{\Lambda}_h = \mathbf{D}_f^{(h)} \boldsymbol{\Lambda}_f \mathbf{D}_f^{*(h)}, \tag{23}$$

$$\mathbf{v}_h^{(g)} = \mathbf{D}_f^{*(h)-1}(\mathbf{v}_f^{(g)} - \mathbf{v}_f^{(h)}), \tag{24}$$

$$\boldsymbol{\Phi}_h^{(g)} = \mathbf{D}_f^{*(h)-1} \boldsymbol{\Phi}_f^{(g)} \mathbf{D}_f^{*(h)-1}, \tag{25}$$

$$\boldsymbol{\psi}_h^{(g)} = \mathbf{D}_f^{(h)} \boldsymbol{\psi}_f^{(g)} \mathbf{D}_f^{(h)}, \tag{26}$$

where the diagonal matrix $\mathbf{D}_f^{*(h)}$ has elements such that the fixed scale-determining elements of $\boldsymbol{\Lambda}_f$ are unchanged.

The model is identified by standardization of the parameters using a reference group. From (22–26) it can be seen that care must be taken regarding the interpretation of these standardized parameters and the testing of various hypotheses regarding them. This is explicated as follows.

Consider different choices of reference group in (22–26). This affects the $\mathbf{D}_f^{(h)}$, $\mathbf{D}_f^{*(h)}$, and $\mathbf{v}_f^{(h)}$ arrays. We note that the standardized parameters of $\mathbf{v}^{(g)}$ $\boldsymbol{\Phi}^{(g)}$ and $\boldsymbol{\psi}^{(g)}$ can only be interpreted in a relative sense, compared over groups. Relationships over groups will be unchanged, although expressed in different metrics, when different reference groups are used. For instance, a test of group-invariant factor mean vectors or factor covariance matrices is unaffected by choice of reference group.

Unless the factor and error covariance matrices are group invariant, the elements of $\mathbf{D}^{(g)}$ and, therefore, $\mathbf{D}^{*(g)}$ are different over groups. Therefore, the relationship between standardized parameters within groups will in general be affected by the choice of reference group. For instance, tests of equality of the standardized factor loadings will in general depend on choice of reference group. For ease of interpretation, it is sometimes useful to present the estimates with the standardization of $\mathbf{D}^{(g)} = \mathbf{I}$, $g = 1, 2, \ldots, G$.

The Special Case of One Factor

It is interesting to look at the assumptions of invariance, (13), (14) and (15), from a different perspective. Consider item i in group g for the special case of a single factor ($k = 1$). Modifying the assumptions of the general model above in a nonsignificant way, we may take the error $\varepsilon_i^{(g)}$ to be normal and independently distributed of $\xi^{(g)}$, obtaining

$$P(x_i^{(g)} = 1 \mid \xi^{(g)}) = \int_{w_i^{(g)}}^{\infty} \phi(z)\, dz, \tag{27}$$

where $w_i^{(g)} = \psi_{ii}^{(g)-1/2}(\tau_i - \lambda_i \times \xi^{(g)})$. We note that (27) is equivalent to the item characteristic curve of the normal ogive model [see e.g. Lord & Novick, 1968]:

$$a_i^{(g)} = \psi_{ii}^{(g)-1/2}\lambda_i, \tag{28}$$

$$b_i^{(g)} = \frac{\tau_i}{\lambda_i}. \tag{29}$$

The slope (discriminating power) parameters $a_i^{(g)}$ of the normal ogive are here confounded with group-specific error variance. The error variance is viewed as representing other, random factors. When such disturbances differ between groups, the assumption of invariant loadings may hold while the assumption of invariant slope parameters does not. [This may be compared to the discussion in Lord & Novick, 1968, p. 353].

In this connection it is instructive to consider the common use of summed item scores as an estimator of $E(\xi) = \nu$, when comparing groups with unequal factor and error variance. Assuming equal measurement parameters over groups and a one-factor model, we obtain the expected value of the sum for each group by (11).

$$E\left(\sum_{i=1}^{p} x_i^{(g)}\right) = \sum_{i=1}^{p} P(x_i^{(g)} = 1) = \sum_{i=1}^{p} \int_{\theta_i^{(g)}}^{\infty} \phi(\xi_i^{**(g)})\, d\xi_i^{**(g)}, \tag{30}$$

where in this case $\theta_i^{(g)} = (\lambda_i^2\phi^{(g)} + \psi_{ii}^{(g)})^{-1/2} \times (\tau_i - \lambda_i \nu^{(g)})$, where $\phi^{(g)}$ is the variance of $\xi^{(g)}$. We note that groups ordered in a certain way with respect to their $\nu^{(g)}$-values need not be ordered in the same way with respect to their values for the expected sum. The effect of unequal factor and error variances may be strong when the threshold values are unevenly distributed around the factor means.

Consider the following example which may not be unrealistic for a set of attitude items. Assume two groups with $\nu^{(1)} = .0$, $\nu^{(2)} = .2$, $\phi^{(1)} = 1.0$, $\phi^{(2)} = .6$ and a set of p equivalent items all with $\tau = .5$ and $\lambda = .7$. Also assume equal error variances within groups, $\psi_{ii}^{(1)} = .5$ and $\psi_{ii}^{(2)} = .2$. The communalities are then 50% and 60% respectively. We obtain

$$E\left(\sum_{i=1}^{p} x_i^{(1)}\right) = .306p \qquad \text{and} \qquad E\left(\sum_{i=1}^{p} x_i^{(2)}\right) = .302p$$

Thus, the existing level difference is concealed and in fact reversed.

Estimation of the Model

To estimate the model we will use the same approach as Muthén [1978]. Information from the first and second order marginal proportions for each group are used to fit the model by generalized least squares (GLS).

From a random sample of individuals we create $\mathbf{t}^{(g)} = (\mathbf{t}_1^{(g)\prime}, \mathbf{t}_2^{(g)\prime})'$ corresponding to the population entity $\boldsymbol{\theta}^{(g)} = (\boldsymbol{\theta}_1^{(g)\prime}, \boldsymbol{\theta}_2^{(g)\prime})'$, $g = 1, 2, \ldots, G$. Let $\mathbf{W}^{(g)}$ be the estimator of the covariance matrix of $\mathbf{t}^{(g)}$ as given in Muthén [1978]. The groups are assumed to be independent, so we obtain the GLS fitting function as the sum

$$F = \frac{1}{2} \sum_{g=1}^{G} (\mathbf{t}^{(g)} - \boldsymbol{\theta}^{(g)})' \mathbf{W}^{(g)-1} (\mathbf{t}^{(g)} - \boldsymbol{\theta}^{(g)}), \tag{31}$$

which will be minimized numerically with respect to the parameters. For the iterative minimization of F we use the Fletcher-Powell method [see Fletcher & Powell, 1963] as modified by Gruvaeus & Jöreskog [Note 1]. This method requires the first derivatives of F.

Let $\mathbf{C}^{(g)} = \partial\boldsymbol{\theta}^{(g)}/\partial\boldsymbol{\pi}'$, where $\boldsymbol{\pi}$ is the vector of the distinct, nonfixed parameters. As in Christoffersson [1975] and Muthén [1978] it follows that the estimated asymptotic covariance matrix of the estimated parameter vector may be obtained as

$$\left[\sum_{g=1}^{G} \mathbf{C}^{(g)\prime} \mathbf{W}^{(g)-1} \mathbf{C}^{(g)} \right]^{-1}. \tag{32}$$

Also, in large samples, twice the minimum value of F will be chi-square distributed with degrees of freedom equal to $G \times p(p + 1)/2$ minus the number of elements in $\boldsymbol{\pi}$ (assuming that p is equal for all groups). This gives a test of the restrictions imposed on the first and second order marginal probabilities.

The calculations involved in the estimation and testing are carried out by the computer program LADI, developed by Muthén and Dahlqvist [Note 2].

Examples

Three examples will be given corresponding to different degrees of invariance over groups. We will study two sets of data from the General Social Survey of the National Opinion Research Center. The first example concerns the response to the following questions on abortion.

"Please tell me whether or not *you* think it should be possible for a pregnant woman to obtain a *legal* abortion if...

A. If there is a strong chance of serious defect in the baby?
B. If she is married and does not want any more children?
C. If the woman's own health is seriously endangered by the pregnancy?
D. If the family has a very low income and cannot afford any more children?
E. If she became pregnant as a result of rape?
F. If she is not married and does not want to marry the man?"

Yes-answers are coded, 1; No-answers, 0. The responses Don't Know and No Answer are discarded.

The factor structure of these items has been studied by Muthén [1981]. We will select three items of special interest, namely items B, D, and F. In the analysis of Muthén [1981] these items were found to represent one factor, which could be interpreted as favoritism of abortion for social rather than medical reasons. These items have a much lower rate of approval than the others. For Protestants the responses are expected to have a strong positive relation to the respondent's education.

We will analyze three independent groups of individuals corresponding to three levels of schooling completed by the respondent: less than high school (LHS), high school only (HS), and more than high school (MHS). Data for Protestants from the survey year 1978 are considered. The sample size of each group is given in Table 1 below.

Since the same items were administered to all groups we assume invariant measurement parameters,

$$\tau^{(1)} = \tau^{(2)} = \tau^{(3)} = \tau, \tag{33}$$

$$\lambda^{(1)} = \lambda^{(2)} = \lambda^{(3)} = \lambda, \tag{34}$$

where λ is the 3×1 vector of loadings. To determine the scale of the factor we will fix one loading to one. For reasons described earlier we will use one group as a reference group, setting

$$\nu^{(f)} = 0, \tag{35}$$

and

$$\psi^{(f)} = \mathbf{I} - \text{diag}(\lambda \phi^{(f)} \lambda'). \tag{36}$$

We take group f to be the respondents with less than high school education. All ψ-matrices

TABLE 1

Estimates for the Abortion Data*

Item	Threshold	Loading	Error Variance
B.	.633 (.072)	1.000** -	.100 -
D.	.386 (.068)	.969 (.031)	.154 -
F.	.574 (.069)	.938 (.032)	.209 -
Group	**Number of observations**	**Factor mean**	**Factor variance**
LHS	316	.000** -	.900 (.043)
HS	315	.242 (.114)	1.641 (.521)
MHS	267	.750 (.147)	2.935 (1.059)

*Standard errors in parentheses.
**Fixed parameters.

are assumed to be diagonal. The fit of this model is good with a chi-square of 1.2 with 2 degrees of freedom and a probability level of .561.

Our next step is to test the more restrictive model with invariant error variances

$$\psi^{(1)} = \psi^{(2)} = \psi^{(3)}. \tag{37}$$

This also yields a good fit with a chi-square of 4.1 with 8 d.f. ($p = .845$). The difference in chi-square is distributed as chi-square and has a value of 3.0 with 6 d.f. Thus, we cannot reject the hypothesis of invariant error variances. The parameter estimates for this model are presented in Table 1. Favoritism of abortion increases with level of schooling. Separate tests of education invariant factor means and factor variances, respectively, resulted in two degree of freedom chi-square differences of 22.6 and 8.7.

The second example also concerns data from NORC's General Social Survey. In 1976 nine items regarding feelings of anomia were administered. The wording of the items is given in Table 2. Affirmative answers are coded as 1 as opposed to 0. The responses Don't Know and No Answer are discarded.

TABLE 2

Nine Anomia Items from NORC's General Social Survey 1976

Item wording
Now I'm going to read you several more statements. Some people agree with a statement, others disagree. As I read each one, tell me whether you more or less <u>agree</u> with it, or more or less <u>disagree</u>.
1. Next to health, money is the most important thing in life.
2. You sometimes can't help wondering whether anything is worthwhile any more.
3. To make money, there are no right and wrong ways any more, only easy ways and hard ways.
Now I'd like your opinions on a number of different things.
4. Nowadays, a person has to live pretty much for today and let tomorrow take care of itself. Do you more or less agree with that, or more or less disagree?
5. In spite of what some people say, the lot (situation/condition) of the average man is getting worse, not better.
6. It's hardly fair to bring a child into the world with the way things look for the future.
Now I'm going to read you several more statements.
7. Most public officials (people in public office) are not really interested in the problems of the average man.
8. These days a person doesn't really know whom he can count on.
9. Most people don't really care what happens to the next fellow.

TABLE 3

Test of the Number of Factors for the Anomia Items

Number of factors	Degrees of freedom	Chi-square value		
		LHS	HS	MHS
		All nine items		
1	27	58.8	72.9	69.9
2	19	24.9	33.5	33.1
3	12	*	*	*
		Dropping items 1 and 3		
1	14	40.0	37.7	35.8
2	8	5.3	5.8	4.5

*No solution with positive error variances obtained.

In sociological studies, feelings of alienation, anomia, and powerlessness are often hypothesized to be associated with a person's socioeconomic status and education. In our example, we will therefore again relate the response to the background variable years of schooling, using the same three categories as in the previous example. We will describe the analysis of this data set in some detail.

As a first step, the number of factors was tested in each of the three education groups separately. For this, the method of Muthén [1978] was used. The results in terms of fit are given in the upper panel of Table 3. For three factors an admissible solution was not obtained for any of the groups. Negative error variances occurred, indicating that the three-factor model was not suitable. Further analyses showed that this problem could be eliminated by the deletion of item 1 and item 3. Both of these items deal with monetary aspects. When these items were dropped, the results in the lower panel of Table 3 were obtained. In this case, a two-factor model fitted well in all three education groups.

The promax-rotated loading matrices had a similar pattern for the groups, with Λ of the type:

$$\Lambda = \begin{array}{c|cc} \textit{Item} & & \\ 2 & x & 0 \\ 4 & x & 0 \\ 5 & x & 0 \\ 6 & x & 0 \\ 7 & 0 & x \\ 8 & 0 & x \\ 9 & 0 & x \end{array} \tag{38}$$

where x's denote large positive loadings and zeros represent small loadings. All groups had a positive factor correlation of about .6. As opposed to Factor 1, Factor 2 has to do with anomia feelings of an *inter*personal type.

Following these exploratory analyses, the hypothesis of invariant measurement pa-

TABLE 4
Estimates for the Anomia Data*

Item	Measurement parameters		
	Threshold	Loadings	
2	-.097	.703	.000**
	(.052)	(.074)	-
4	-.240	.604	.000**
	(.046)	(.073)	-
5	-.635	.842	.000**
	(.062)	(.081)	-
6	-.359	1.000**	.000**
	(.057)	-	-
7	-.656	.000**	.728
	(.057)	-	(.065)
8	-.956	.000**	1.000**
	(.069)	-	-
9	-.627	.000**	.868
	(.064)	-	(.112)
	LHS	HS	MHS
Number of observations	436	453	410
Mean of Factor 1	.000**	-.514	-.875
	-	(.075)	(.133)
Mean of Factor 2	.000**	-.622	-.738
	-	(.175)	(.119)
Variance of Factor 1	.766	.417	.346
	(.078)	(.190)	(.169)
Variance of Factor 2	.693	.153	.147
	(.097)	(.170)	(.163)
Factor covariance	.476	.175	.140
	(.061)	(.107)	(.086)
Error variance: item 2	.621	.447	.582
	-	(.222)	(.278)
" 4	.720	.596	.420
	-	(.351)	(.249)
" 5	.457	.099	.119
	-	(.052)	(.073)
" 6	.234	.274	.114
	-	(.151)	(.068)
" 7	.633	.154	.224
	-	(.183)	(.268)
" 8	.307	.054	.119
	-	(.062)	(.133)
" 9	.479	.092	.045
	-	(.125)	(.063)

*Standard errors in parentheses.
**Fixed parameter.

rameters was tried, making possible the estimation of group differences in factor means, factor variances, and the factor correlation. Here, the small loadings in Λ of (38) were fixed to zero, hypothesizing a simple measurement structure.

This model resulted in a chi-square of 44.3 with 45 degrees of freedom and a probability level of .503. Adding the assumption of invariant error variances gave a chi-square of 70.0 with 59 degrees of freedom. The difference is 25.7, which with 14 degrees of freedom is significant on the 5%-level. The first model, allowing for differing error variances, was chosen. The estimates are given in Table 4. Here, the group with least schooling (LHS) is chosen as the base for comparison. We note that the estimated factor means decrease with increasing number of years of schooling. From the loadings we find that a low value of each factor is associated with a low propensity to agree with the items, that is a low degree of anomia feeling. The relationship with education is very similar for the two factors. For both factors the difference is larger between LHS and HS than between HS and MHS. As discussed above, a test of quality of the factor means within each of the groups is not permissible, because the factors are not expressed in the same metric.

The estimates in Table 4 also suggest that the HS and MHS groups are more homo-

TABLE 5
Item Wording for Seven "Interpersonal Relations" (IR) Items and Four "Neurotic Illness" (NI) Items

IR Items

Now please think about all the people in your life who live in or near (this town). This includes the people you live with, your family and friends.

1. Among your family and friends, how many people are there who are immediately available to you whom you can talk with frankly, without having to watch what you say?
(Response alternatives: None, 1-2, 3-5, 6-10, 11-15, more than 15
-Who is this mainly?)

2. Now I would like to ask you if there is anyone who lives in or near (this town) who knows you very well as a person (this includes friends as well as family members).
(Response alternatives: No-one, Yes (qualified), Yes
-Who is this?)

3. Is there any particular person you feel you can lean on?
(Response alternatives: No-one, Yes-but don't need anyone, Yes
-What is his/her name?)

4. Do you feel there is one particular person who feels very close to you?
(Response alternatives: No-one, Not sure, Yes
-Who is this mainly?)

5. When you are happy, is there any particularly person you can share it with -- someone who you feel sure will feel happy simply because you are?
(Response alternatives: No-one, Yes
-Who is this mainly?)

6. At present, do you have someone you can share your most private feelings with (confide in) or not?
(Response alternatives: Same as for item 5)

7. Are there ever times when you are comforted by being held in someone's arms or not?
(Response alternatives: No, Yes
-By whom mainly?)

NI Items

In the last month, have you suffered from any of the following?

8. Anxiety (Response alternatives: Yes, No)
9. Depression (")
10. Irritability (")
11. Nervousness (")

TABLE 6
Standardized Estimates for Seven Interpersonal Relations Items and Four Neurotic Illness Items
(Chi-square = 102.8, d.f. = 88, p = .134)*

	Males (N = 266)				Females (N = 318)			
Item	Threshold	Loadings		Error variance	Threshold	Loadings		Error Variance
1	-.293	1.000**	.000**	.527	-.028	1.000**	.000**	.558
	(.071)	-	-	-	(.067)	-	-	-
2	-.237	.753	.000**	.732	-.151	.993	.000**	.564
	(.068)	(.102)	-	-	(.066)	(.103)	-	-
3	-.497	1.160	.000**	.363	-.640	1.208	.000**	.355
	(.073)	(.110)	-	-	(.071)	(.106)	-	-
4	-.936	.990	.000**	.536	-.551	1.128	.000**	.438
	(.079)	(.121)	-	-	(.071)	(.113)	-	-
5	-.942	1.273	.000**	.233	-.672	1.368	.000**	.172
	(.079)	(.124)	-	-	(.070)	(.111)	-	-
6	-.640	1.325	.000**	.169	-.469	1.183	.000**	.381
	(.077)	(.116)	-	-	(.068)	(.110)	-	-
7	-.338	.848	.000**	.660	-.900	1.020	.000**	.540
	(.070)	(.108)	-	-	(.078)	(.135)	-	-
8	.737	.000**	1.000**	.328	.696	.000**	1.000**	.390
	(.066)	-	-	-	(.071)	-	-	-
9	.864	.000**	1.161	.094	.768	.000**	1.031	.352
	(.077)	-	(.070)	-	(.073)	-	(.096)	-
10	.568	.000**	.850	.514	.188	.000**	.854	.555
	(.072)	-	(.078)	-	(.067)	-	(.096)	-
11	.856	.000**	1.143	.122	.840	.000**	1.074	.297
	(.074)	-	(.068)	-	(.074)	-	(.100)	-
				Factor covariance matrix				
		.473				.442		
		(.072)				(.066)		
		-.114	.672			-.044	.610	
		(.050)	(.071)			(.038)	(.081)	
				Factor correlation				
		-.202				-.085		

*Standard errors in parentheses.
**Fixed parameter.

geneous with respect to the factor variances and covariance, compared to the LHS groups. However, testing invariance of the factor covariance matrix over all three education groups resulted in a chi-square difference value of 7.5. With six degrees of freedom this hypothesis cannot be rejected.

The third and final example concerns data for all the married persons ($N = 584$) from a sample of adults of Canberra, Australia interviewed in 1977. These data have been kindly provided by Dr. Paul Duncan-Jones. Two sets of items are considered. The first concerns seven facets of "interpersonal relationships" (IR), where the respondents are scored 1 if they have this kind of relationship with someone and the main provider of that kind of relationship is their husband or wife. The second set of items concerns four indicators of "neurotic illness" (NI). The wording of these items is given in Table 5.

Anticipating sex differences, the analysis was performed with males ($N = 266$) and females ($N = 318$) in two separate groups. A simple structure two-factor model was hypothesized with the IR items loading on one factor and the NI items loading on the other [lambda pattern similar to that in (38)]. The factors were allowed to be correlated. This model fit well for both males and females. With 43 degrees of freedom the chi-square was 51.3 ($p = .180$) for males and 47.9 ($p = .281$) for females.

In a simultaneous analysis of the two groups, the hypothesis of measurement parameter invariance over sex (for thresholds and loadings) was rejected. With 93 degrees of freedom the chi-square was 146.1 ($p = .000$). Simultaneous, two-group analyses for each set of items separately (single-factor models), resulted in a chi-square of 61.3 with 33 degrees of freedom ($p = .002$) for the IR items and a chi-square of 5.3 with 6 degrees of freedom ($p = .512$) for the NI items. For the set of IR items this suggests a sex specific reaction to the measurement instrument.

Returning to the simple structure two-factor model for both sets of items, a model with partial measurement invariance was finally tried. This only restricted the thresholds and loadings for the NI items to be sex invariant. Since the IR factor is sex specific it is not possible to identify an IR factor mean sex difference or a sex difference for the error variances of the IR items, as was previously discussed. For these items we may use the (single group) restrictions of a zero factor mean and unit variances for the latent response variables. This resulted in a well-fitting model with a chi-square value of 102.8 with 88 degrees of freedom ($p = .134$). The model does allow for a comparison over sex of the NI factor mean and variance. Females had a larger factor mean value (a stronger tendency to admit to these type of symptons), estimated to .635 as compared to zero for males. Females also had a smaller factor variance, estimated to .013 as compared to .672 for males. Separate tests of sex invariant factor means and variances, respectively, resulted in significant one degree of freedom chi-square differences of 15.2 and 14.6.

In Table 6 are given the estimates from the simultaneous two-group analysis (allowing sex specific NI factor means and variances). In this case estimates are given with the standardization in each group of zero factor means and unit variances for all the latent response variables.

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