British Journal of Mathematical and Statistical Psychology (1990) 43, 131–143 *Printed in Great Britain*

Moments of the censored and truncated bivariate normal distribution

Bengt Muthén†

Graduate School of Education, University of California, Los Angeles, Los Angeles, California 90024, USA

Published results on the moments of censored and truncated bivariate normal distributions do not include explicit formulas for all combinations of limits in a form that is readily adapted for computation. Moments for truncation and censoring that can take place both from above and below in both variables are given in a general form from which special cases are easily obtained. The attenuation of the correlation coefficients is studied in a series of graphs and related to examples of factor analysis.

1. Introduction

This paper is motivated by the following type of measurement problems common in the analysis of behavioural data. In a study of depression and anxiety, Muthén (1989*a*) analysed dichotomously scored symptom data obtained by questionnaire. Here, 0 and 1 correspond to absence and presence of the symptom, respectively. Since the data were obtained from a normal population, 1s were rather rare observations. A factor analysis was carried out for 3161 individuals on 41 such skewed, dichotomously scored items using tetrachoric correlation coefficients, and several well-defined factors were identified. The tetrachoric assumption of underlying normality for continuous latent response variables was tested by means of the method of Muthén & Hofacker (1988) and could not be rejected. As described in Muthén (1989*a*), this means that the strong skewness in the dichotomous items can be interpreted as arising from underlying normal symptom variables which have thresholds far out in the right tail of their distributions. For simplicity in further analyses, it is of interest to sum such dichotomous variables over items loading on each factor. This yields observed variables that have a large percentage of cases with value zero. Given underlying normality for the dichotomous components and a large number of components, a rough approximation is to postulate that each such summed score represents a normal variable which is censored from below at the value zero. Analogous to the situation of tetrachoric correlations and phi coefficients (see, for example, Lord & Novick, 1968), it is then of interest to relate the correlation between the underlying normal variables to the correlation between the observed, censored variables. We may also be interested in the correlation between observed variables for individuals who exhibit none of the symptoms of the scores involved. This leads to considering correlations for truncated variables, in this case with truncation point zero.

Assume, for example, two summed scores, Depression and Anxiety for which the

†Requests for reprints.

Table 1. Factor analysis model

Variable	Factor I	II	III
1	0.887	0	0
2	0.827	0	0
3	0	0.941	0
4	0	0.733	0
5	0	0	0.782
6	0	0	0.837

	Factor correlations I	II	III
I	1		
II	0.501	1	
III	0.733	0.477	1

	Observed variable correlation matrix 1	2	3	4	5	6
1	1					
2	0.734	1				
3	0.418	0.390	1			
4	0.326	0.304	0.690	1		
5	0.509	0.475	0.351	0.274	1	
6	0.544	0.508	0.376	0.293	0.654	1

correlation coefficient is to be determined. In Muthén (1989*a*), data from two different sites were analysed and it was found that different symptom (factor) levels were present. As a first motivating, although artificial example, consider a Depression score and an Anxiety score for each of the two sites, where in both cases censoring takes place from below at zero. Assume two underlying bivariate normal variables with correlation 0.5 and variances one in both sites, with means 0.39, 0.39 for the first site and means 0.84, 0.84 for the second site. While the underlying correlation is 0.5 in both sites, it is clear that the ordinary correlation between the observed, censored, or truncated variables will differ across sites due to different degrees of censoring across sites, where the percentages are about 65 per cent for both variables in the first site and about 80 per cent for both variables in the other site. The resulting correlations between the observed variables will be developed in Sections 2–4 below and we will return to this example in the concluding discussion section.

With several censored or truncated variables, a factor analysis may be of interest. Consider as a second example the confirmatory factor analysis model of Table 1. As we will elaborate further in the concluding Section 5, the model and the parameter values are derived from analyses based on Muthén (1989*a*). There is a simple structure loading matrix for six observed variables and there are three correlated factors. Assuming residual variances that yield unit observed variable variances, this model gives the correlation matrix at the bottom of Table 1.

In the real data set on which this example is based, there was strong censoring with censoring percentages for the six variables of 0.79, 0.85, 0.87, 0.97 and 0.98. Assume now that the six variables structure of Table 1 pertains to uncensored normal variables. Using the formulas to be derived below, we can find the correlations among the corresponding observed, censored variables and consider these for factor analysis. We will return to this example in the concluding section. As we will see, factor analysis of the observed variable correlations distorts the structure for the underlying uncensored variables of Table 1, yielding biased factor analysis parameter estimates. We argue that it is more realistic that a simple structure exists in this way for underlying variables and that this solution gets distorted when analysing censored, or truncated, versions of these underlying variables. This is because the distributional assumptions of the standard linear factor analysis model for observed censored, or truncated variables, such as residuals being uncorrelated with factors, probably do not hold (cf. regression analysis with such dependent variables, as in for instance, Maddala, 1983).

The main aim of this paper is to review and derive results for the moments of the censored standard normal distribution as they pertain to results for correlation coefficients. This is carried out in Sections 2–4. Section 5 returns to the motivating examples of the present section and uses the derived results to demonstrate some practical consequences. Issues of estimation of correlations and factor analysis thereof are also included.

As a first step of obtaining the moments of the censored standard bivariate normal distribution we consider the moments of the truncated bivariate normal distribution. Rosenbaum (1961) gave explicit formulas for truncation from below in both variables, $b_1 < y_1 < \infty$, $b_2 < y_2 < \infty$ by direct integration, while Tallis (1961) gave general formulas for multivariate truncation from below in the multivariate normal distribution using the moment-generating function. Generalizing results by Aitkin (1964), Regier & Hamdan (1971) expressed the moments in terms of Hermite–Chebyshev polynomials for truncation from below and above in both variables. An explicit formula was, however, only given for the case of truncation from below at the same point in both variables. Des Raj (1953) gave the moments for truncation and censoring from below and above in one of the variables, while Shah & Parikh (1964) gave recurrence relations among the moments in various cases of truncation. In none of the above cases, however, are the explicit moment formulas readily apparent for all cases of truncation from below and/or above in the two variables.

In this paper we use the moment-generating approach of Tallis (1961) to give explicit moment formulas for truncation in the standard bivariate normal distribution. To cover in a single formula truncation that is either from below or from above in each of the variables, we consider truncation of the more general form $b_1 < y_1 < a_1$, $b_2 < y_2 < a_2$. Building on this, we also give the desired explicit moment formulas for the corresponding censored case. The formulas are directly amenable to computation. Several plots are given to describe the attenuation of the correlation coefficient in some pertinent truncated and censored cases.

2. Truncated case

Consider the bivariate normal probability

$$\pi(a_1, b_1, a_2, b_2) = \pi = \int_{b_1}^{a_1} \int_{b_2}^{a_2} \phi(y_1, y_2)\, dy_2\, dy_1, \tag{1}$$

where we use $\phi(\cdot)$ to denote the standard univariate or bivariate normal density. Similar to Tallis (1961), equation (2), we may express the moment generating function (see, for example, Hoel, Port & Stone, 1971) M as

$$\pi M = e^{-1/2\mathbf{t}'\mathbf{R}\mathbf{t}} \times \int_{b_1^*}^{a_1^*} \int_{b_2^*}^{a_2^*} \phi(y_1, y_2)\, dy_2\, dy_1 = A \times B, \tag{2}$$

say, where $\mathbf{t}' = (t_1 t_2)$ is a vector of real numbers, $a_i^* = a_i - (t_i + \rho t_j)$, $b_i^* = b_i - (t_i + \rho t_j)$, ρ is the correlation coefficient of a 2×2 matrix $\mathbf{R}$, and $i = 1$ or 2, with j assuming the opposite value. Using obvious notation, let

$$\phi(y_i) \int_{b_j^*}^{a_j^*} \phi(y_j | y_i)\, dy_j = f(t_i, y_i), \tag{3}$$

so that

$$\delta B/\delta t_i = \delta \int_{b_i^*}^{a_i^*} f(t_i, y_i)\, dy_i/\delta t_i$$

$$= \phi(a_i^*) \int_{b_j^*}^{a_j^*} \phi(y_j | a_i^*)\, \delta a_i^*/\delta t_i - \phi(b_i^*) \int_{b_j^*}^{a_j^*} \phi(y_j | b_i^*)\, \delta b_i^*/\delta t_i^*$$

$$+ \int_{b_i^*}^{a_i^*} \delta f(t_i, y_i)/\delta t_i\, dy_i. \tag{4}$$

Considering $\pi\, \delta M/\delta t_i$ at $t_i = 0$ we have with $c = (1 - \rho^2)^{-1/2}$, the univariate expectation

$$\begin{aligned} \pi E(y_i; a_1, b_1, a_2, b_2) = {} & -\phi(a_i)\, [\Phi[(a_j - \rho a_i)c] - \Phi[(b_j - \rho a_i)c]] \\ & + \phi(b_i)\, [\Phi[(a_j - \rho b_i)c] - \Phi[(b_j - \rho b_i)c]] \\ & - \rho\phi(a_j)\, [\Phi[(a_i - \rho a_j)c] - \Phi[(b_i - \rho a_j)c]] \\ & + \rho\phi(b_j)\, [\Phi[(a_i - \rho b_j)c] - \Phi[(b_i - \rho b_j)c]], \end{aligned} \tag{5}$$

where $\Phi(\cdot)$ is the univariate standard normal distribution function. When any one of the as and bs equals $\pm\infty$, (5) simplifies by noting that $\phi(\pm\infty) = 0$, $\Phi[\infty] = 1$, $\Phi(-\infty) = 0$. For example, when $a_1 = \infty$, $a_2 = \infty$, (5) gives the result for truncation from below in both variables given by Rosenbaum (1961). When $b_1 = -\infty$, $b_2 = -\infty$ we have truncation from above in both variables,

$$\pi(a_1, -\infty, a_2, -\infty)E(y_i; a_1, -\infty, a_2, -\infty) = -\phi(a_i)\Phi[(a_j - \rho a_i)c] - \rho\phi(a_j)\Phi[(a_i - \rho a_j)c]. \tag{6}$$

When $a_1 = \infty$, $b_2 = -\infty$, we have truncation from below in y_1 and from above in y_2,

$$\pi(\infty, b_1, a_2, -\infty)E(y_1; \infty, b_1, a_2, -\infty) = \phi(b_1)\Phi[(a_2 - \rho b_1)c] - \rho\phi(a_2)\,[1 - \Phi[(b_1 - \rho a_2)c]]. \tag{7}$$

$$\pi(\infty, b_1, a_2, -\infty)E(y_2; \infty, b_1, a_2, -\infty) = -\phi(a_2)[1 - \Phi[(b_1 - \rho a_2)c]] + \rho\phi(b_1)\Phi[(a_2 - \rho b_1)c]]. \tag{8}$$

Furthermore, $\pi\,\delta^2 M/\delta t_j\,\delta t_i$ at $t_i = 0$, $t_j = 0$ yields the expectation

$$\begin{aligned}
\pi E(y_i y_j; a_1, b_1, a_2, b_2) = {} & \begin{Bmatrix}1\\ \rho\end{Bmatrix}\pi - \begin{Bmatrix}1\\ \rho\end{Bmatrix} a_i\phi(a_i)\,[\Phi[(a_j - \rho a_i)c] - \Phi[(b_j - \rho a_i)c]] \\
& + \begin{Bmatrix}0\\ c^{-1}\end{Bmatrix}\phi(a_i)\,[\phi[(a_j - \rho a_i)c] - \phi[(b_j - \rho a_i)c]] \\
& + \begin{Bmatrix}1\\ \rho\end{Bmatrix} b_i\phi(b_i)\,[\Phi[(a_j - \rho b_i)c] - \Phi[(b_j - \rho b_i)c]] \\
& - \begin{Bmatrix}0\\ c^{-1}\end{Bmatrix}\phi(b_i)\,[\phi[(a_j - \rho b_i)c] - \phi[(b_j - \rho b_i)c]] \\
& + \begin{Bmatrix}\rho\\ 1\end{Bmatrix}\rho a_j\phi(a_j)\,[\Phi[(a_i - \rho a_j)c] - \Phi[(b_i - \rho a_j)c]] \\
& + \begin{Bmatrix}c^{-1}\\ 0\end{Bmatrix}\rho\phi(a_j)\,[\phi[(a_i - \rho a_j)c] - \phi[(b_i - \rho a_j)c]] \\
& + \begin{Bmatrix}\rho\\ 1\end{Bmatrix}\rho b_j\phi(b_j)\,[\Phi[(a_i - \rho b_j)c] - \Phi[(b_i - \rho b_j)c]] \\
& - \begin{Bmatrix}c^{-1}\\ 0\end{Bmatrix}\rho\phi(b_j)\,[\phi[(a_j - \rho b_j)c] - \phi[(b_i - \rho b_j)c]].
\end{aligned} \tag{9}$$

Here, $\{c_{ii}/c_{ji}\}$ gives the multiplying factor for $i=j$ (c_{ii}) and $i\neq j$ (c_{ji}), respectively.

Special cases are again obtained by letting certain truncation points go to $\pm\infty$, where we note that the product $x\phi(x)\to 0$ when $x\to\pm\infty$. When $a_1=\infty$, $a_2=\infty$, the results again agree with those of Rosenbaum (1961), while truncation from above in both variables ($b_1=-\infty$, $b_2=-\infty$) yields

$$\begin{aligned}\pi(a_1,-\infty,a_2,-\infty)E(y_i^2;a_1,-\infty,a_2,-\infty)=\pi(a_1,-\infty,a_2,-\infty)-a_i\phi(a_i)\Phi[(a_j-\rho a_i)c]\\ -\rho^2 a_j\phi(a_j)\Phi[(a_i-\rho a_j)c]\\ +c^{-1}\rho\phi(a_j)\Phi[(a_i-\rho a_j)c],\end{aligned} \tag{10}$$

$$\begin{aligned}\pi(a_1,-\infty,a_2,-\infty)E(y_1y_2;a_1,-\infty,a_2,-\infty)=\rho\pi(a_1,-\infty,a_2,-\infty)\\ -\rho a_1\phi(a_1)\Phi[(a_2-\rho a_1)c]\\ +c^{-1}\phi(a_1)\phi[(a_2-\rho a_1)c]\\ -\rho a_2\phi(a_2)\Phi[(a_1-\rho a_2)c].\end{aligned} \tag{11}$$

With truncation from below in y_1 and above in y_2 ($a_1=\infty$, $b_2=\infty$) we have

$$\begin{aligned}\pi(\infty,b_1,a_2,-\infty)E(y_1^2;\infty,b_1,a_2,-\infty)=\pi(\infty,b_1,a_2,-\infty)\\ +b_1\phi(b_1)\Phi[(a_2-\rho b_1)c]\\ -\rho^2a_2\phi(a_2)\,[1-\Phi[(b_1-\rho a_2)c]]\\ -c^{-1}\rho\phi(a_2)\phi[(b_1-\rho a_2)c],\end{aligned} \tag{12}$$

$$\begin{aligned}\pi(\infty,b_1,a_2,-\infty)E(y_2^2;\infty,b_1,a_2,-\infty)=\pi(\infty,b_1,a_2,-\infty)\\ -a_2\phi(a_2)\,[1-\Phi[(b_1-\rho a_2)c]\\ -\rho^2b_1\phi(b_1)\Phi[(a_2-\rho b_1)c]\\ -c^{-1}\rho\phi(b_1)\phi[(a_2-\rho b_1)c]],\end{aligned} \tag{13}$$

$$\begin{aligned}\pi(\infty,b_1,a_2,-\infty)E(y_1y_2;\infty,b_1,a_2,-\infty)=\rho\pi(\infty,b_1,a_2,-\infty)\\ +\rho b_1\phi(b_1)\Phi[(a_2-\rho b_1)c]\\ -c^{-1}\phi(b_1)\phi[(a_2-\rho b_1)c]\\ -\rho a_2\phi(a_2)\,[1-\Phi[(b_1-\rho a_2)c]].\end{aligned} \tag{14}$$

3. Censoring

Consider now a standard bivariate normal distribution with censoring,

$$y_i=a_i, \quad \text{if} \quad y_i \geqq a_i$$

$$y_i=b_i, \quad \text{if} \quad y_i \leqq b_i. \tag{15}$$

The two-dimensional y_2, y_1-plane is then divided into nine areas. We must recognize that under censoring, as opposed to truncation, we also have contributions to the expectation from units outside the area

$$b_1 < y_1 < a_1,$$

$$b_2 < y_2 < a_2, \tag{16}$$

since the unit is then not discarded. As seen below, both variables are censored in four areas. In four other areas one variable is censored and the other is not, whereupon it remains to determine the marginal expectation of the latter. In each of the nine areas, the expectation is obtained from various special cases of the general formulas (5) and (9), where censoring points go to $\pm\infty$.

Let $E(y_1^r y_2^s)$ represent the four r, s combinations 1, 0 ($E(y_1)$), 2, 0 ($E(y_1^2)$), 0, 1 ($E(y_2)$), 0, 2 ($E(y_2^2)$), and 1, 1 ($E(y_1 y_2)$). Summing over the nine areas, we have

$$\begin{aligned}
E(y_1^r y_2^s) = {} & \pi(b_1, -\infty, \infty, a_2) b_1^r a_2^s \\
& + \pi(a_1, b_1, \infty, a_2) E(y_1^r; a_1, b_1, \infty, a_2) a_2^s \\
& + \pi(\infty, a_1, \infty, a_2) a_1^r a_2^s \\
& + \pi(b_1, -\infty, a_2, b_2) b_1^r E(y_2^s; b_1, -\infty, a_2, b_2) \\
& + \pi(a_1, b_1, a_2, b_2) E(y_1^r y_2^s); a_1, b_1, a_2, b_2) \\
& + \pi(\infty, a_1, a_2, b_2) a_1^r E(y_2^s; \infty, a_1, a_2, b_2) \\
& + \pi(b_1, -\infty, b_2, -\infty) b_1^r b_2^s \\
& + \pi(a_1, b_1, b_2, -\infty) E(y_1^r; a_1, b_1, b_2, -\infty) b_2^s \\
& + \pi(\infty, a_1, b_2, -\infty) a_1^r b_2^s.
\end{aligned} \tag{17}$$

As in the truncated case, various special cases of censoring are obtained by letting censoring points go to $\pm\infty$. In doing so, it is clear that certain of the nine areas will vanish and not contribute to (17).

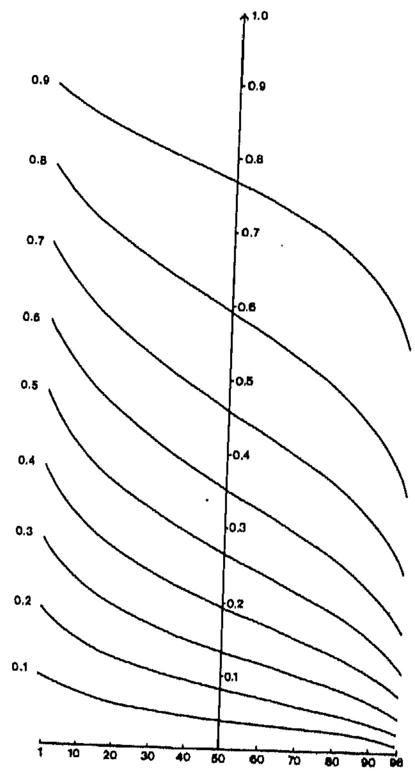

Figure 1. Correlation attenuation for truncation from below in both variables. x axis is percentage truncation; y axis is orginal correlation value.

4. Behaviour of the correlation coefficient

The above formulas will now be used to graphically describe the relationship between the correlation in various truncated and censored distributions with that of the correlation in the original standard bivariate normal distribution. The bivariate normal probabilities involved were calculated using a routine described by Kirk (1973); however, see also Divgi (1979). A listing of a FORTRAN program used by the author for these computations may be obtained on request.

Figure 1 describes the correlation in the case of truncation from below at the same point in both variables ($a_1 = \infty$, $b_1 = b$, $a_1 = \infty$, $b_2 = b$); this figure was also given in Regier & Hamdan (1971). In Fig. 2 the corresponding graphs are given for the censored case. We note that the attenuation is considerably smaller in the censored case.

In Figs 3 and 4 the original correlation is held constant at 0.5, while the limits of the two variables are allowed to differ. These figures describe the correlation in the case of truncation and censoring both from below in both variables ($a_1 = \infty$, b_1, $a_2 = \infty$, b_2), and from below in one and from above in the other ($a_1 = \infty$, b_1, a_2, $b_2 = -\infty$).

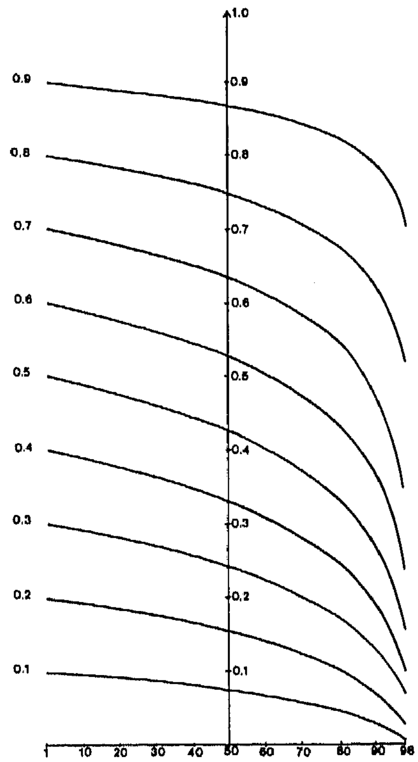

Figure 2. Correlation attenuation for censoring from below in both variables. x axis is percentage censoring; y axis is original correlation value.

We may compare the results of Figs 1 and 2 with those of Figs 3 and 4. For instance, Fig. 1 shows that equal truncation from below of about 58 per cent in variables correlated 0.5 yields an attenuated correlation of 0.25. This agrees with the 'BB 0.25' curve of Fig. 3, which also gives unequal truncation points yielding this attenuation. It is interesting to note that when truncation takes place instead from below in one variable and from above in the other, an attenuated correlation of 0.25 is obtained already with approximately 23 per cent truncation in each variable (see 'BA 0.25' curve in Fig. 3). Similar results are found in Fig. 4. For instance, while equal censoring from below of about 65 per cent in both variables gives an attenuated correlation of 0.4, a similar amount of censoring from below in one variable and from above in the other gives an attenuated correlation of only 0.25.

5. Discussion

Let us now return to the two motivating examples of the introductory section. In the first example, the underlying correlation was 0.5 for each of two sites. The degree of censoring was 65 per cent for the two variables in the first site and 80 per cent for the two variables in the other site. We may use Figs 3 and 4 to roughly obtain the corresponding truncated and censored variable correlations. They are lower and

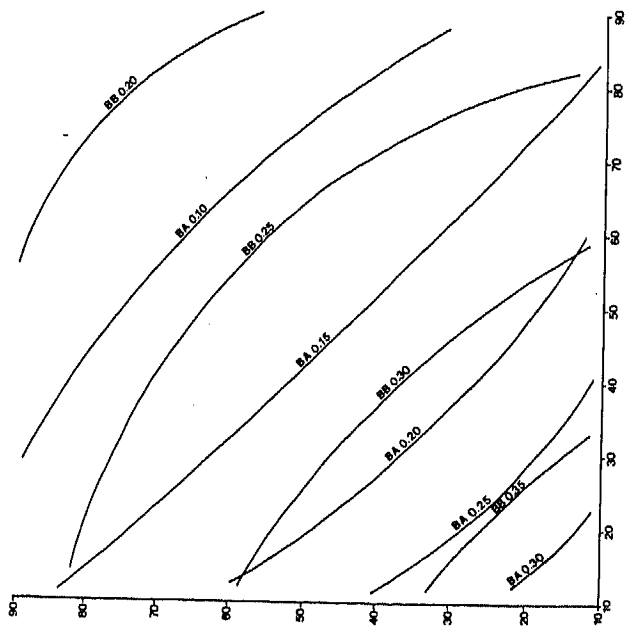

Figure 3. Correlation attenuation for truncation from below or above with original correlation 0.5. x axis is percentage truncation; y axis is percentage truncation. BB x, truncation from below in both variables resulting in attenuated correlation x; BA x, truncation from below in one variable and truncation from above in the other resulting in attenuated correlation x.

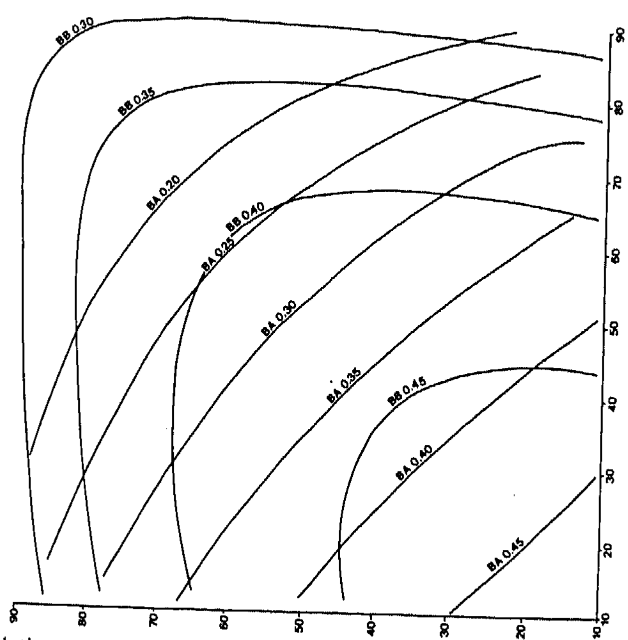

Figure 4. Correlation attenuation for censoring from below or above with original correlation 0.5. x axis is percentage censoring; y axis is percentage censoring. BB x, censoring from below in both variables resulting in attenuated correlation x; BA x, censoring from below in one variable and censoring from above in the other resulting in attenuated correlation x.

Table 2. Factor analysis for censored/truncated variables

Variable	Factor I	Factor II	Factor III
1	0.806/0.677	0/0	0/0
2	0.708/0.573	0/0	0/0
3	0/0	0.824/0.670	0/0
4	0/0	0.601/0.476	0/0
5	0/0	0/0	0.563/0.438
6	0/0	0/0	0.528/0.470

Factor correlations[a]

	I	II	III
I	1	0.300	0.544
II	0.358	1	0.277
III	0.491	0.255	1

Observed variable correlation matrix[a]

	1	2	3	4	5	6
1	1	0.388	0.136	0.096	0.161	0.174
2	0.571	1	0.116	0.083	0.136	0.146
3	0.239	0.206	1	0.319	0.082	0.087
4	0.176	0.151	0.495	1	0.059	0.061
5	0.222	0.197	0.119	0.084	1	0.206
6	0.207	0.187	0.112	0.077	0.297	1

[a]Censored/truncated variable correlations are given in the lower/upper-triangular part.

different as expected, about 0.25 and 0.20 for truncation and about 0.40 and 0.35 for censoring. We would draw the incorrect conclusion of low and unequal correlations.

The second example considered factor analysis. Using the specifications described in connection with Table 1, we obtain the corresponding censored and truncated variable correlations as given at the bottom of Table 2. This table also gives the estimates of loadings and factor correlations obtained by maximum likelihood factor analysis of these correlations. While the model for the original variables fits perfectly to the correlations of Table 1, the same model applied to the censored and truncated variables fits remarkably well but not perfectly, with 6 degree of freedom chi-square test of fit values of 0.34 and 0.03, respectively, for a sample size of 3161. Comparing Tables 1 and 2, it is seen that the correlations are severely attenuated by censoring/truncation and that although much of the attenuation gets absorbed into deflated loading estimates, there is also a considerable amount of misestimation of the factor correlations. The model distortion may go unnoticed since it cannot be detected by the chi-square model test of fit.

The model and parameter values used in Table 1 were obtained from a real-data analysis based on the depression and anxiety study of Muthén (1989*a*), corresponding to the three factors Anxious Depression, Phobic Anxiety, and Somatic Anxiety. Muthén (1989*b*) proposes new methodology for both computation of sample correlations underlying censored variables and factor analysis of these correlations.

The correlation estimates are obtained by maximum likelihood (ML) in two steps for each pair of variables, first estimating means and variances by two univariate ML analyses, and then estimating the correlation coefficient by bivariate ML holding the values for the means and the variances fixed. This procedure yields relatively simple and robust computations and the estimates are usually very close to the full (bivariate) information ML estimates. The factor analysis is then carried out by weighted least squares using as weight matrix an estimate of the asymptotic covariance matrix of the estimated correlations (Muthén, 1987, 1989*b*).

The factor analysis procedure of Muthén (1989*b*) resulted in the estimated parameter values of Table 1 with a chi-square model test of fit of 6.62 with 6 degrees of freedom (sample size 3161). Hence, the model fits extremely well. It may be noted that ML factor analysis of the ordinary Pearson product moment correlations gave a considerably higher chi-square value of 28.47 with the same degrees of freedom.

A further example of the use of the formulas derived in this paper was given in Muthén, Kaplan & Hollis (1987) related to selectively missing data.

Acknowledgements

This research was supported by grant no. SES-8312583 from the National Science Foundation. I wish to thank Chuen-Rong Chan for drawing the figures.

References

Aitkin, M. A. (1964). Correlation in a singly truncated bivariate normal distribution. *Psychometrika*, **29**, 263–270.

Des Raj (1953). On estimating the parameters of bivariate normal populations from doubly and singly linearly truncated samples. *Sankhya*, **12**, 277–290.

Divgi, D. R. (1979). Calculation of the tetrachoric correlation coefficient, *Psychometrika*, **44**, 169–172.

Kirk, D. B. (1973). On the numerical approximation of the bivariate normal (tetrachoric) correlation coefficient. *Psychometrika*, **38**, 259–268.

Hoel, P. G., Port, S. C. & Stone, C. J. (1971). *Introduction to Probability Theory*. New York: Houghton Mifflin.

Lord, F. M. & Novick, H. (1968). *Statistical Theories of Mental Test Scores*. Reading, MA: Addison-Wesley.

Maddala, G. S. (1983). *Limited-dependent and Qualitative Variables in Econometrics*. Cambridge, UK: Cambridge University Press.

Muthén, B. (1987). *LISCOMP. Analysis of Linear Structural Equations with a Comprehensive Measurement Model, User's Guide*. Mooresville, IN: Scientific Software, Inc.

Muthén, B. (1989*a*). Dichotomous factor analysis of symptom data. In W. W. Eaton & G. Bohrnstedt (Eds), *Latent Variable Models for Dichotomous Outcomes: Analysis of Data from the Epidemiological Catchment Area Program. Sociological Methods and Research*, **18**, 19–65.

Muthén, B. (1989*b*). Tobit factor analysis. *British Journal of Mathematical and Statistical Psychology*, **42**, 241–250.

Muthén, B. & Hofacker, C. (1988). Testing the assumptions underlying tetrachoric correlations. *Psychometrika*, **53**, 563–578.

Muthén, B., Kaplan, D. & Hollis, M. (1987). On structural equation modeling with data that are not missing completely at random. *Psychometrika*, **42**, 431–462.

Regier, M. H. & Hamdan, M. A. (1971). Correlation in a bivariate normal distribution with truncation in both variables. *Australian Journal of Statistics*, **13**(2), 77–82.

Rosenbaum, S. (1961). Moments of a truncated bivariate normal distribution. *Journal of the Royal Statistical Society, Series B*, **23**, 223–229.

Shah, S. M. & Parikh, N. T. (1964). Moments of single and doubly truncated standard bivariate normal distribution. *Vidya* (*Gujarat University*), **7**, 82–91.
Tallis, G. M. (1961). The moment generating function of the truncated multinormal distribution, *Journal of the Royal Statistical Society, Series B*, **23**, 223–229.

Received 13 May 1987; revised version received 31 August 1988